ON THE USE OF THE TOTAL SCALAR POTENTIAL IN
THE NUMERICAL SOLUTION OF FIELD PROBLEMS IN
ELECTROMAGNETICS

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SUMMARY

The paper summarizes the formulation of a set of computer algorithms for the solution of the three-dimensional non-linear Poisson field problem. Results are presented that were obtained by applying algorithms to the analysis of two-dimensional magnetostatic fields. Scalar and vector potentials were used, and it is shown that the convenient single valued scalar potential associated with the induced sources gives severe accuracy problems in permeable regions. The results become as good as those obtained using vector potential if the scalar potential associated with the total field is used for permeable regions. The combination of two scalar potentials has a significant advantage for three-dimensional problems.

INTRODUCTION

The non-linear Poisson equation occurs in many areas of physics and engineering. The equation is relatively easy to solve compared to other defining equations, but for many applications the solutions must be very accurate. In magnetostatic problems, for example, the geometry of boundaries and surfaces separating differing media is often complicated and field accuracies of the order of 0.1 per cent and higher are essential. These conditions are frequently encountered in the wide range of electromagnets associated with the design of charged particle accelerators, spectrometers, detectors, focusing devices and plasma containment experiments used in physics and also in the broad spectrum of machines, transformers, etc. used in electrical engineering. Although the methods discussed in this paper are of general applicability they are looked at with particular reference to electromagnetics.

Both differential and integral operator formulations have been used to solve the magnetostatic problem. Many well established programs solve the two-dimensional cases using differential formulations based directly on the defining equation usually in terms of the single component vector potential.1-3 These programs are capable of giving high accuracy although the position of the far field boundary can have a significant effect on the results. This latter difficulty has been overcome at the expense of increased computational cost by solving the integral form of the equations in terms of field components directly. An added advantage of this approach is that only regions containing material media (e.g., iron) are discretised. This is very useful in extending to three dimensions where the need to have a mesh of elements connecting many different regions of complex shape is seen as a limitation of the differential approach. This difficulty is avoided in programs based on integral formulations4-6 and at the present time these provide a general technique for solving non-linear three-dimensional magnetostatic problems providing high accuracy is not required.
Increasing the order of discretisation above a moderate level does not help since computing times escalate rapidly because of the fully populated matrix associated with integral operators. In this respect the use of the differential operator is to be preferred and hence the recent work reported by Zienkiewicz et al., on the coupling of differential and integral techniques, is very relevant. In any such scheme, differential, integral or coupled, the choice of independent variable is important and to ensure maximum economy a scalar potential is to be preferred having only one unknown for each element node point.

The question arises then, which type of potential to use? Many investigators have used the reduced scalar potential, the gradient of which is defined to be the field from the magnetized regions of the problem, that is to say, the total field diminished by the known source fields. Because this potential is single valued everywhere it is very convenient to use and in order to assess the possible accuracy and economy that would be achieved a differential operator magnetostatic code was written. The program was designed so that integral techniques could be coupled to the differential formulation. Close examination of the results for analytic test cases showed that in the regions containing permeable media (e.g. iron) the answers were sometimes very inaccurate. It was realized that this was caused by the cancellation of the known source and calculated potential fields. Since the difference between these quantities is small the error in the fields from the calculated potential dominates. It was clear that the difference quantity should be calculated directly and this implies a formulation in terms of a scalar potential that represents the total field.

The scalar potential from current sources is multi-valued and in conductors is undefined, which means that the two types of formulation must be combined. In regions containing currents a reduced scalar potential can be used and elsewhere a total scalar potential. The Galerkin procedure was used and the two potentials are linked at the interface between the two regions by applying the usual normal and tangential continuity condition. The magnitude of the potential jump is obtained entirely from a line integral of the source field. It will be demonstrated that by using this procedure good results can be obtained everywhere and that reliable fields can be computed interior to saturable iron regions, thus enabling non-linear problems to be solved. Good results are also obtained with a vector potential formulation, which does not suffer from cancellation problems interior to the iron, however, as already stated, there are 3 unknowns per point in three dimensions.

MAGNETIC FIELD EQUATIONS AND POTENTIALS

Basic equations for magnetostatics

The basic equations of magnetostatics are given below in terms of the magnetic induction \( \mathbf{B} \), the field intensity \( \mathbf{H} \), and the current density \( \mathbf{J} \). For a full derivation of these relationships see Reference 11.

\[
\begin{align*}
\text{div } \mathbf{B} &= 0 \quad (1) \\
\text{curl } \mathbf{H} &= \mathbf{J} \quad (2)
\end{align*}
\]

These are connected by the constitutive relation

\[
\mathbf{B} = \mu_0 \mu (|\mathbf{H}|) \mathbf{H} \quad (3)
\]

or by the magnetization \( \mathbf{M} \)

\[
\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H} \quad (4)
\]
or
\[ \mathbf{M} = \chi \mathbf{H} = (\mu - 1) \mathbf{H} \]  
(5)

Where \( \mu \) and \( \chi \) are the permeability and susceptibility respectively and can be non-linear functions of the Field Intensity \( \mathbf{H} \) for permeable materials. Also \( \mu_0 \) is the permeability of free space.

Practical magnetic field problems usually involve current sources so that it is sometimes convenient to express the total field \( \mathbf{H} \) as the sum of the field for the sources \( \mathbf{H}_S \) and the field from the induced magnetization \( \mathbf{H}_M \) viz:
\[ \mathbf{H} = \mathbf{H}_M + \mathbf{H}_S \]  
(6)

where
\[ \mathbf{H}_M = \frac{1}{4\pi} \nabla \int_{\Omega_i} \mathbf{M}(r') \cdot \nabla \left( \frac{1}{|r - r'|} \right) d\Omega_j \]  
(7)

and
\[ \mathbf{H}_S = \frac{1}{4\pi} \int_{\Omega_j} \mathbf{J} \times \left( \frac{r - r'}{|r - r'|} \right) d\Omega_j \]  
(8)

i.e.
\[ \text{curl } \mathbf{H}_S = \mathbf{J} \]  
(9)

where \( r \) and \( r' \) are the position vectors of the field and source points respectively and the integration regions are over the volume of all iron and current regions respectively. In general the current density \( \mathbf{J} \) can be specified exactly and \( \mathbf{H}_S \) in equation (8) can be evaluated analytically or by quadrature.

From the above equations various integral and differential equations can be derived which can then form the basis of numerical algorithms. E.g. from equations (6), (7) and (5)
\[ \mathbf{M}(r) = \chi(r) \left[ \mathbf{H}_M(r) - \frac{1}{4\pi} \nabla \int_{\Omega_i} \mathbf{M}(r') \cdot \nabla \left( \frac{1}{|r - r'|} \right) d\Omega_j \right] \]  
(10)

is the integral equation in \( \mathbf{M} \) as solved in the GFUN program.

Various other equations can be derived by introduction of suitable potentials. Reduced scalar potential \( \phi \)

From equations (2), (6) and (9) curl \( \mathbf{H}_M = 0 \) therefore a scalar potential can be introduced
\[ \mathbf{H}_M = -\nabla \phi \]  
(11)

and from equations (1) and (3)
\[ \text{div } \frac{\mathbf{B}}{\mu_0} = -\text{div } \mu \nabla \phi + \text{div } \mu \mathbf{H}_S \]

or
\[ \nabla \cdot \mu \nabla \phi = \text{div } \mu \mathbf{H}_S \]  
(12)

which is a non-linear PDE of the Poisson type. This equation is subject to boundary conditions
\[ \phi \to 0 \text{ at infinity} \]  
(13)
At the interface between regions of different material the normal component of $B$ and the tangential component of $H$ must be continuous, i.e. between two regions of different permeability $\mu_1$ and $\mu_2$ the first condition gives

$$\mu_1 \left( \frac{\partial \phi}{\partial n} \right)_1 - \mu_2 \left( \frac{\partial \phi}{\partial n} \right)_2 = (\mu_1 - \mu_2) H_{s n}$$

(14)

where $H_{s n}$ is the normal component of the source field $H_s$ which is identical for both regions and $n$ is the normal direction.

The second condition is automatically satisfied for any continuous potential $\phi$. The potential is unique over all space including conductor regions and has been used as the basis of finite element algorithms both directly in differential form and in the corresponding integral form. However the choice of this potential can lead to numerical difficulties.

For example, inside the iron

$$B = \mu \mu_0 H = \mu \mu_0 (H_s - \nabla \phi)$$

and in this expression the computed quantity is $\phi$, the gradient of which can be of the same order as $H_s$. (In the limit $\mu \rightarrow \infty$, $H \rightarrow 0$) Thus there exists the possibility of large errors occurring in $H$. This is demonstrated later in an actual problem. It is safer to compute the small difference quantity therefore directly and this can be arranged by choosing a different potential, as shown in the next section.

**Total scalar potential $\psi$ combined with reduced potential $\phi$**

From equation (2) for simply connected, regions it can be seen that curl $H = 0$ if current carrying regions are excluded. For these regions therefore introduce a scalar potential $\psi$ which defines the field

$$H = -\nabla \psi$$

(15)

From equations (1) and (3)

$$\text{div } B = -\nabla \cdot \mu \mu_0 \nabla \psi = 0$$

(16)

which is a non-linear PDE of the Laplace type.

In order to introduce the current carrying regions it is necessary to introduce either suitable 'cuts' to ensure a single valued potential or to arrange separate regions utilizing both potentials $\phi$ and $\psi$. In the latter case a particularly simple scheme may be devised. Consider the example shown in Figure 1.

Here two regions are shown; region 1 has no conductors present and could be a region of permeable iron whereas region 2 has conductors and could be the space extending to infinity. This situation embraces a very large class of practical problems.

Clearly the field in region 1 can be represented by the total scalar potential $\psi$ and region 2 by the reduced scalar potential $\phi$, i.e. in region 1 the PDE is given by

$$\nabla \cdot \mu \nabla \psi = 0$$

and from equation (12) the PDE for region (2) is given by

$$\nabla \cdot \mu \nabla \phi = \text{div } \mu H_s$$

† Multiply connected regions, e.g. transformer problems, can also be handled easily by taking suitable 'cuts',
furthermore if region 2 is a region of constant permeability or free space then the PDE reduces to, as in the problem under consideration

\[ \nabla^2 \phi = 0 \quad \text{(simple Laplace)} \]

There is of course the possibility of a discontinuity of the two potentials at the interface between the two regions which follows from the interface conditions, e.g. at point P in Figure 2.

\begin{align*}
-\mu_1 \frac{\partial \psi}{\partial n} &= \mu_2 \left( -\frac{\partial \phi}{\partial n} + H_{sn} \right) \\
-\frac{\partial \psi}{\partial t} &= \left( -\frac{\partial \phi}{\partial t} + H_{st} \right)
\end{align*}
where \( \mathbf{H}_\text{t} \) is the tangential component of \( \mathbf{H} \) (18) can be integrated along the interface to give

\[
\psi = \phi - \int \mathbf{H}_\text{t} \, dt
\]  

(19)

Clearly by Amperes Law and the definition of the Potential \( \psi \) [equation (18)]

\[
\oint \mathbf{H}_\text{t} \, dt = 0
\]

where the integration is taken along any closed path contained in region 1. If at a convenient point take \( \psi = \phi \) then equation (19) defines the potential jump at any point on the surface.

**Vector potentials**

The simplest potential for two-dimensional problems is the vector potential \( \mathbf{A} \) for which a wide variety of codes exist. 1-3 The definition of \( \mathbf{A} \) is

\[
curl \mathbf{A} = \mathbf{B}
\]

Thus automatically satisfying equation (1).

\[
\text{div} \mathbf{A} = 0
\]

In 2D \( \mathbf{A} \) has only one component normal to the plane considered given by \( \mathbf{A} \) and satisfies the PDE.

\[
\nabla \cdot \frac{1}{\mu \mu_0} \nabla \mathbf{A} = \mathbf{J} \quad \text{(where} \ \mathbf{J} \ \text{is the appropriate component of current)}
\]  

(20)

In 3D \( \mathbf{A} \) has 3 components and so is less attractive. In some circumstances it may not be desirable to include the current sources so one may define a reduced vector potential \( \mathbf{A}_\text{M} \) by

\[
\mathbf{A} = \mathbf{A}_\mu + \mathbf{A}_s
\]  

(21)

and

\[
curl \frac{1}{\mu} \nabla \mathbf{A}_s = \mathbf{J}
\]  

(22)

and \( \mathbf{A}_\text{M} \) satisfies

\[
\nabla \cdot \frac{1}{\mu} \nabla \mathbf{A}_\mu = \nabla \cdot (1 - \frac{1}{\mu}) \mathbf{A}_s
\]  

(23)

Furthermore one may combine \( \mathbf{A} \) and \( \mathbf{A}_\text{M} \) in a manner analogous to the previous use of \( \psi \) and \( \phi \) but this will not be pursued here.

**Summary of the potentials**

The table below summarizes the various potentials and PDE’s used in magnetic calculation.

<table>
<thead>
<tr>
<th>Field</th>
<th>Potential</th>
<th>PDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \mathbf{H} = \mathbf{H}_\text{M} + \mathbf{H}<em>S ) ( \mathbf{H}</em>\text{M} = -\nabla \phi )</td>
<td>( \phi )</td>
<td>( \nabla \cdot \mu \nabla \phi = \text{div} \mu \mathbf{H}_S )</td>
</tr>
<tr>
<td>2. ( \mathbf{H} = -\nabla \psi )</td>
<td>( \psi )</td>
<td>( \nabla \cdot \mu \nabla \psi = 0 )</td>
</tr>
<tr>
<td>3. ( \mathbf{B} = \text{Curl} \mathbf{A} )</td>
<td>( \mathbf{A} )</td>
<td>( \nabla \cdot \frac{1}{\mu_0 \mu} \nabla \mathbf{A} = \mathbf{J} )</td>
</tr>
<tr>
<td>4. ( \mathbf{A} = \mathbf{A}_\text{M} + \mathbf{A}_S ) ( \text{curl} \frac{1}{\mu_0} \text{curl} \mathbf{A}_S = \mathbf{J} )</td>
<td>( \mathbf{A}_\text{M} )</td>
<td>( \nabla \cdot \frac{1}{\mu} \nabla \mathbf{A}_\text{M} = \nabla \cdot \left( 1 - \frac{1}{\mu} \right) \mathbf{A}_S )</td>
</tr>
</tbody>
</table>
As can be seen all the PDE’s are of the general form

\[ \nabla \cdot k \nabla u = Q \]  

(24)

Some useful integral forms of (24) are given in the next section.

**Integral equations from Green’s theorem**

In certain circumstances integral equations offer some advantages over PDE’s and it is convenient at this stage to quote some general expressions.

For a field satisfying the general form (24) the potential \( u \) at a point interior to a region can be expressed by use of Green’s second theorem by integrals of \( u \) on the boundary and over the interior.

\[
4\pi k(r)u(r) = \int k(r') \left( \frac{1}{|r - r'|} \frac{\partial u(r')}{\partial n} - \frac{\partial}{\partial n} \left( \frac{1}{|r - r'|} \right) \right) dS
\]

\[ + \int_{\Omega} \nabla \cdot k(r') \nabla \left( \frac{1}{|r - r'|} \right) \left( \frac{1}{|r - r'|} \right) Q d\Omega \]  

(25)

For a region of constant \( k \) the volume integrals vanish resulting in an integral equation over the boundaries.

i.e.

\[
4\pi u = \int \frac{\partial u}{\partial n} dS - \int \left( \frac{1}{r_i} \right) u dS
\]  

(26)

where \( r_i \) is the distance between source and field point. For a given problem (26) has been solved by a wide variety of methods.15,16,13,14

**Which potential?**

In order to gain some experience in the effectiveness of the various formulations the 4 potentials considered here have been compared for a range of two-dimensional problems. The numerical methods used will be outlined in the next section.

**NUMERICAL METHODS**

**General comments**

For an efficient three-dimensional technique certain objectives can be defined.

1. Scalar potential instead of a vector or field for dependent variable, i.e. 1 unknown per point rather than 3.
2. Differential operator for non-linear regions. One needs a mesh of volume elements for non-linear regions in any case so a differential scheme offers the greatest economy – sparse system.
3. Integral operator for linear regions. One needs a mesh or elements on surfaces only – particularly efficient for external infinite regions.

The prime objective was to establish the viability of scalar potential and to this end experimental codes were written solving both differential and integral equations. In the sections that follow very brief indications will be given of the numerical methods used.
Differential operators

Standard methods of finite elements are used to develop systems of equations for the solution of equation (24). It is convenient to approach the problem by way of Weighted Residuals using the Galerkin procedure. Thus the residual for equation (24) is given by:

\[ R = \int_{\Omega} W_i (\nabla \cdot k \nabla u - Q) d\Omega \]  

(27)

for a suitably chosen set of weighting functions \( W_i \). To allow the introduction of local basis functions for finite elements without the restriction of 2nd order continuity on \( u \) the operator part of (27) is transformed by use of Green's Theorem (integrating by parts) i.e.:

\[ R = \int_{\Omega} k \nabla W_i \cdot \nabla u d\Omega - \int_{\Omega} W_i Q d\Omega - \int_{\partial \Omega} W_i \frac{\partial u}{\partial n} dS \]  

(28)

In some circumstances it may be necessary to transform the 2nd integral as well (Potential type 1 for example).

The Galerkin procedure is introduced by subdividing the domain into a number of finite elements and choosing a set of basis functions such that for the \( n \)th element:

\[ u = N_i u_i + N_j u_j + N_k u_k \]  

(29)

where \( u_i \) etc. are nodal values of \( u \), \( N_i \) are the shape functions and setting

\[ W_i = N_i(x, y, z) \]  

(30)

equation (28) becomes for the \( n \)th element

\[ \int_{\Omega_n} k \nabla N_i \cdot \nabla u d\Omega_n - \int_{\Omega_n} N_i Q d\Omega_n - \int_{\partial \Omega_n} N_i \frac{\partial u}{\partial n} dS_n \]  

(31)

or in matrix form:

\[ Au = F \]

The finite element equations for each potential are now considered in turn.

Scalar potential \( \phi \)

For this potential from equation (12)

\[ Q = \nabla \cdot \mu \mathbf{H}_s \]

In this case it is convenient to transform \( Q \) by Green's Theorem, so equation (27) becomes

\[ R = \int_{\Omega} \mu (\nabla W_i \cdot \nabla \phi - \nabla W_i \cdot \mathbf{H}_s) d\Omega + \int_{\partial \Omega} \mu W_i \left( H_{sn} - \frac{\partial \phi}{\partial n} \right) dS = 0 \]  

(32)

Application of Galerkin to (32) results in finite element equations of the form:

\[ k_i \phi_i + f_i = 0 \]  

(33)
where

\[ k_{ij} = \int_{\Omega} \mu \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) d\Omega \]  

(34)

\[ f_i = -\int_{\Omega} \mu \left( \frac{\partial N_i}{\partial x} H_{Sx} + \frac{\partial N_i}{\partial y} H_{Sy} \right) d\Omega \]  

(35)

\[ q_i = \int_S \mu N_i \left( H_{S\phi} - \frac{\partial \phi}{\partial n} \right) dS \]  

(36)

for the nth element.

The surface integral is applied over the far field boundary and is not required when \( \phi \) is specified.

The above relations are readily extended to three dimensions. For non-linear problems the resulting set of global (merged) equations can be solved iteratively.

*Double scalar potential \( \psi \phi \)*

Here the simply connected problem is divided into regions as defined in Figure 1, and Galerkin weighting is applied to both regions. Region 1 is in general non-linear and region 2 is free space (\( \mu = 1 \)) with conductors.

In each region a finite element subdivision is made and a standard assembly procedure followed.

From equation (28)

\[ R_i = \int_{\Omega_1} \mu \nabla W_i \cdot \nabla \psi d\Omega_1 - \int_{S} \mu W_i \frac{\partial \psi}{\partial n} dS \]

\[ R_2 = \int_{\Omega_2} \mu \nabla W_i \cdot \nabla \phi d\Omega_2 - \int_{S} \mu W_i \frac{\partial \phi}{\partial n} dS \]

hence:

\[ R = R_1 + R_2 = 0 \]

This results on using the interface condition (17) in

\[ \int_{\Omega_1} \mu \nabla W_i \cdot \nabla \psi d\Omega_1 + \int_{\Omega_2} \mu \nabla W_i \cdot \nabla \phi d\Omega_2 = -\int_{S} \left( -\mu \frac{\partial \psi}{\partial n} + \frac{\partial \phi}{\partial n} \right) dS = -\int_{S} W_i H_{S\phi} dS \]  

(37)

After applying Galerkin to (37) the coefficient matrix is identical to (34) with appropriate permeability.

The RHS are now zero everywhere except on the interface. This means that \( H_{S\phi} \) is only needed at these points. At the interface between regions 1 and 2 either \( \phi \) or \( \psi \) can be eliminated by use of equation (19) viz:

\[ \psi = \phi - \int H_{S\phi} dt \]

Hence for an element containing an interface segment the contribution is:

\[ F = Kg - h \]  

(38)
where $K$ is the element matrix (i.e. equation (34)) and $g$ and $h$ are column vectors whose members are given by:

\[
\begin{align*}
    g_i &= \int_0^1 H_{si} dt \\
    h_i &= \int_0^1 N_i H_{si} dt
\end{align*}
\]

See Figure 3.

Vector potentials

For total $A$ the matrix equations are given by equations (28), (31) and (34) with $k$ set to $1/\mu$. The right hand side of the equations depends only on the conductor elements and is

\[
f_i = \int_{\Omega_i} N_i J d\Omega_i
\]
For $A_M$ the coefficient matrix is the same as for $A$; however the right hand sides are non-zero everywhere except in the regions where $\mu \neq 1$, i.e. in a manner similar to $\phi$

\[
R = \gamma \left[ \nabla \cdot \nabla A_M - \nabla W \left( \frac{\gamma_0 - \gamma}{\gamma} \right) \nabla A_M \right] d\Omega + \int_S \gamma N_e \left( \frac{\gamma_0 - \gamma}{\gamma} \right) \nabla A_n - \frac{\partial A_n}{\partial N_e} dS \quad (41)
\]

with corresponding finite element equations analogous to (34)-(36). Where $\gamma = 1/\mu$.

Computer programs

A single computer program has been written to embrace the four potential types discussed. Linear and quadratic triangular elements have been used and full details of explicit forms for the coefficient matrices etc. can be found in standard texts dealing with the finite element method.\textsuperscript{17}

The results for the test problems used are presented in the next section.

Integral operators

Returning briefly to integral equations it appears that for linear regions (constant $k$ or $\mu$) these offer an attractive alternative. Equation (26) has been solved and applied to numerous problems by many investigators.\textsuperscript{15-17} In particular a general purpose two- and three-dimensional computer program called BIM has been written.\textsuperscript{13,18} All of the potential types discussed are available in the BIM program. There are of course many alternative formulations and those are dealt with in great detail in the literature.\textsuperscript{7} Results for the selected test problems using BIM are also presented in the following section for comparison.

Of considerable interest in the present context is the possibility proposed by Zienkiewicz of combining the differential and integral approaches.\textsuperscript{7} Work is now in progress using $\psi$ for the internal non-linear regions in differential form and $\phi$ for external linear regions in integral form.

RESULTS

A 2D finite element program with a pre- and post-processor has been developed to test the viability of the various potentials described in section ‘Magnetic field equations and potentials’. The data for the analysis program is generated by an interactive program that uses graphics for checking data and displaying results.

The geometry is defined as a set of simply shaped regions – quadrilaterals that can have surfaces with non-zero curvature. The regions are automatically meshed with a specified subdivision and each region has associated mirror and rotational reflections. The reflection conditions make it possible to describe a periodic structure by only giving data for its unit cell. The method of defining the geometry of a problem is efficient; only 13 instructions were needed to generate the motor shown in Figure 8.

Results for two shielding problems

(a) The fields were computed for a hollow iron cylinder of constant permeability, submerged in a constant external field. The dimensions of the problem were:

Inside radius of the hollow cylinder 0·1 m
Outside radius of the hollow cylinder 0·2 m
Far field boundary defined at a radius of 4 m
Permeability of the cylinder 1,000
Constant external field of 1·0 T.
The fields at a radius of 0·15 m are shown in Figure 4(a). The exact answer is compared to those obtained using reduced scalar and vector potentials and the two scalar potentials. The field at the centre of the hollow cylinder is shown in the figure (B₀).

For this particular problem the cancellation between the source and induced fields does not lead to large errors. This is because the source field is constant in magnitude and direction over all space and the linear or quadratic shape functions for the variation of the potential are a good approximation to the analytic solution.

(b) The analogous problem of shielding a bifilar conductor by use of an external hollow iron cylinder provides a simple analytic test for a case in which the source field is not constant. In fact the source field has a dipole distribution and the results presented in Figure 4(b) show large errors for reduced scalar potential compared to the other potentials at the same level of discretisation. However it is to be expected that the results for reduced scalar potential will improve as the number of elements is increased but at a greater computational cost.

Figure 4(a). Modulus of the field $|B|$ in iron as a function of azimuth for a hollow iron cylinder in a constant external field. Problem discretised into 1,000 nodes (1st order)
Results for a C magnet

In the simple C magnet shown in Figure 5 the fields in the iron showed very large errors caused by cancellation of $H_m$ and $H_z$ when a reduced scalar potential was used. The results for various potentials at a moderate level of discretisation (1,000 nodes, 1,500 elements) are shown in Figure 6(a). As the number of elements is increased the accuracy of the reduced scalar potential solution should improve. The results given in Figure 6(b) demonstrate convergence towards the correct solution when second order elements were used. Table I also shows that the Boundary Integral Method gave poor accuracy when a reduced scalar potential was used. Figure 7 shows a contour map of flux computed using a reduced vector potential solution. The dimensions of the problem were:

- Inside radius of the iron ring 0.05 m
- Outside radius of the iron ring 0.08 m
- Air gap extending from -200 to +200

Table I gives a complete set of results for this magnet.
Figure 5. Geometry of the C magnet. Only the iron elements and conductors are displayed. The external region is similarly discretised out to 0.8 m

Comments on the results

As a guide to the usefulness of the finite element program an example of a real problem is shown in Figure 8, here the flux distribution is shown for a non-linear solution for a reluctance motor. This was another case that showed large errors using reduced scalar potential.

Figure 6(a). Modulus of the field $|\mathbf{B}|$ in iron as a function of azimuth for the C-magnet shown in figure 5. Problem discretised into 1000 nodes with 1st order elements
The results show clearly that the two potential method or vector potential methods are far better than the reduced scalar potential method. The results from GFUN are not very accurate. In the iron of the C magnet near the interface, this is due to the occurrence of an eigenvalue solution (looping) and is one of the problems encountered in using GFUN with large values of permeability.
Table I. Results for the C magnet shown in Figure 5. Modulus of the flux density (|B|) in Tesla, at a radius of 7 cm, as a function of angle. Results for finite element, BIM and GFUN are given. Case A – solution with 1000 nodes, first order elements.
Case B–solution with 3200 nodes, second order elements.

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>FE differential operator</th>
<th>BIM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reduced scalar potential</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Two scalar potentials</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>20</td>
<td>0.94</td>
<td>0.92</td>
</tr>
<tr>
<td>40</td>
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Figure 8. Stepping motor flux plot
CONCLUSIONS

A non-linear 2D computer program has been developed to look at possible solution schemes that could be used for a new 3D program. The code solves for reduced scalar, reduced vector, total vector and a combination of total and reduced scalar potentials. Triangular first and second order elements are available. The results obtained using this program have shown that the reduced scalar potential is of limited use because of the loss of accuracy that occurs inside permeable regions especially when the source field has a gradient. Furthermore this trouble with cancellation between independently computed fields, can also give rise to problems with the reduced vector potential formulation if fields near to interfaces between regions of different permeability are required. The total vector potential of course gives good results but in three dimensions requires three unknowns per mesh point. The most interesting case was found to be the two potential formulation $\psi \phi$. This seems capable of giving reasonable precision everywhere, although cancellation between $H_m$ and $H_i$ can still occur in the free space ($\phi$) regions but this is not so great a difficulty.

The use of total scalar potential in non-linear permeable regions simplifies the defining equation (quasi harmonic) and the source fields enter only as ‘loads’ on the interfaces. It should be stressed that the total potential region can be extended into free space as far as the envelope of the conductors if required.

Future work must investigate methods for improving the accuracy of gradients (fields) obtained from potentials. It may be desirable to use integral operator for regions where high accuracy is needed. In any case it is intended to implement a Boundary Integral scheme for the external region.

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REFERENCES


